Oriented, equivariant $K$-theory and the Sullivan splittings
Kenneth Hansen

This is part 1 of my Ph.D.-thesis, which I wrote at Aarhus University, Matematisk Institut, in 1991.
It has appeared in Matematisk Instituts Preprint Series, no. 35, 1991.

The two other parts are
- The $K$-localizations of Some Classifying Spaces
- The Equivariant $K$-localization of the $G$-Sphere Spectrum
The purpose of this paper is two-fold: To study oriented \(G\)-bundles and the corresponding \(K\)-theory, and to generalize the \(p\)-local splittings

\[ F/O \cong BSO \times \text{Cok} J \quad \text{and} \quad SF \cong J \times \text{Cok} J \]

of Sullivan to the equivariant case.

This paper is divided into 5 parts. We start by recapitulating some essential facts about complex and real \(K\)-theory, and we study their classifying spaces.

In section 2 we introduce \(G\)-\(SO\)-bundles and \(G\)-\(Spin\)-bundles, and we find a connection between these and the 'equivariant Stiefel-Whitney-classes'.

In section 3 we study the space \(BSO_G\) in detail, at least when \(G\) is of odd order. Results about the \(\lambda\)-ring-structure on \(BSO_G\) of Atiyah-Tall and Atiyah-Segal are generalized – here it is necessary to assume that \(G\) is a \(p\)-group, where \(p\) is an odd prime, and that we are in the \(p\)-local situation.

In section 4 we study the space \(S\!F_G\) using the equivariant Adams' conjecture, and finally in section 5 we define the \(e\)-invariant and prove the Sullivan splittings.

Throughout the paper \(G\) is assumed to be finite. All \(G\)-Spaces are assumed to have a basepoint fixed under the \(G\)-action, and normally we consider only \(G\)-\(CW\)-complexes, which are finite and \(G\)-connected.

I would like to thank Ib Madsen, Jørgen Tornehave and Marcel Bökstedt for many enlightening discussions.

1. Preliminary remarks about \(K_G\) - and \(KO_G\) -theory

In this section we briefly describe the functors \(\overline{K}_G(-)\) and \(\overline{KO}_G(-)\) and the corresponding classifying spaces.

\(K_G(X)\) is defined in [S68], p.132, as the Grothendieck group of the additive semigroup of complex \(G\)-bundles over the \(G\)-\(CW\)-complex \(X\). The tensor-product of \(G\)-bundles gives a multiplication on \(K_G(X)\), and \(K_G(X)\) becomes a commutative ring. Similarly, we have the ring \(KO_G(X)\), obtained by using real rather than complex \(G\)-bundles.

\(\overline{K}_G(X)\) is the reduced version of \(K_G(X)\). It is defined as the subgroup of \(K_G(X)\) generated by differences \(E - F\) of complex \(G\)-bundles, such that for every \(x \in X\), the fibres \(E_x\) and \(F_x\) over \(x\) are equivalent \(\mathbb{C}G_x\)-modules. Here \(G_x = \{g \in G \mid gx = x\}\) is the isotropy group.

We define \(\overline{KO}_G(X)\), the reduced version of \(KO_G(X)\), in the same way.

\textbf{Remark 1.1} 
If \(X\) is a \(G\)-connected \(G\)-\(CW\)-complex, i.e. for every subgroup \(H\) of \(G\) the fixed point space \(X^H\) is connected, then it follows from the local triviality condition of [L],
p. 258, that the difference \( E - F \) of \( G \)-bundles over \( X \) is in \( \overline{K}_G(X) \) if and only if the fibres \( E_x \) and \( F_x \) over the basepoint * are isomorphic \( G \)-modules.

By using an equivariant version of Brown’s representation theorem, cf. [LMS], (1.5.11), we see that the functors \( \overline{K}_G(-) \) and \( \overline{KO}_G(-) \) are representable. We denote the classifying spaces by \( BU_G \) and \( BO_G \), respectively.

**Proposition 1.2**

Let \( U_1, \ldots, U_m \) be a complete set of inequivalent, irreducible complex representations of \( G \). Then

\[
(BU_G)^G \cong \prod_{i=1}^m BU
\]

**Proof:**

From [S68], (2.2), we recall the isomorphism

\[
(1.3) \quad \mu : R(G) \otimes K(X) \to K_G(X)
\]

where \( X \) is a trivial \( G \)-Space. As \( \overline{K}_G(X) \cong [X, BU_G]^G \cong [X, BU_G^G] \), and \( R(G) \) is a free \( \mathbb{Z} \)-module generated by \( U_1, \ldots, U_m \), the reduced version of this isomorphism

\[
(1.4) \quad \mu : R(G) \otimes \overline{K}(X) \to \overline{K}_G(X)
\]

would imply the result.

\[
\mu \text{ maps } R(G) \otimes \overline{K}(X) \text{ into } \overline{K}_G(X) : \text{Let } E \text{ and } F \text{ be bundles over } X \text{ with } E - F \in \overline{K}(X). \text{ Then the fibres } E_x \text{ and } F_x \text{ for every } x \in X \text{ have the same dimension. If } V \text{ is a complex } G\text{-representation, then } \mu(V \otimes (E - F)) \text{ is contained in } \overline{K}_G(X) \text{, as the fibres } V \otimes E_x \text{ and } V \otimes F_x \text{ over } x \text{ are isomorphic } \mathbb{C}_G\text{-modules.}
\]

On the other hand, \( \mu(R(G) \otimes \overline{K}(X)) = \overline{K}_G(X) : \text{Let } \xi \in \overline{K}_G(X). \text{ In virtue of (1.3) we can find elements } \xi_1, \ldots, \xi_m \text{ in } K(X), \text{ such that } \xi = \sum_{i=1}^m U_i \otimes \xi_i. \text{ The fibre of the virtual } G\text{-bundle } \xi \text{ over } x \text{ is then, as an element of } R(G), \text{ given by}
\]

\[
\xi_x = \sum_{i=0}^m U_i \otimes (\xi_i)_x = \sum_{i=0}^m d_i \cdot U_i
\]

where \( d_i \) is the complex dimension of \( (\xi_i)_x \). As \( \xi \in \overline{K}_G(X), \xi_x \text{ vanishes as an element of } R(G) \), and we conclude that the \( d_i \)'s are zero. Thus, the \( \xi_i \)'s are contained in \( \overline{K}(X) \), and (1.4) follows.

QED

In the real case we have the following:
**Proposition 1.5**

Let $U_1, U_2, ..., U_k$ be $\mathbb{R}G$-modules, $V_1, V_2, ..., V_m$ be $\mathbb{C}G$-modules, and $W_1, W_2, ..., W_n$ be $\mathbb{H}G$-modules, such that (2.6) in [K] is satisfied. Then

$$BO_G^G = \prod_{x=1}^{k} BO \times \prod_{y=1}^{m} BU \times \prod_{z=1}^{n} BSp$$

**Proof:**

The proof is analogous to that of (1.2) and uses as input, the isomorphism

$$(1.6) \quad \Phi : \bigoplus_{x=1}^{k} KO(X) \oplus \bigoplus_{y=1}^{m} K(X) \oplus \bigoplus_{z=1}^{n} KSp(X) \to KO_G(X)$$

of [K] (5.1). Here $X$ is assumed to be a trivial $G$-space.

QED

Direct sum of vector-bundles makes $\overline{K}_G(-)$ and $\overline{KO}_G(-)$ into Abelian groups, and we thus get an 'additive' $G$-Hopf-space structures on $BU_G$ and $BO_G$.

($G$-Hopf-spaces are defined in [Br], p.II.10.) We denote $BU_G$ and $BO_G$ with this 'additive' structure by $BU_G^\oplus$ and $BO_G^\oplus$.

It is also possible to define 'multiplicative' $G$-Hopf-structures on $BU_G$ and $BO_G$.

For a finite $G$-CW-complex $X$ we consider the sets $1 + \overline{K}_G(X)$ and $1 + \overline{KO}_G(X)$. As every element in $\overline{K}_G(X)$ and $\overline{KO}_G(X)$ is nilpotent, cf. [S68], (5.1), the tensor-product makes $1 + \overline{K}_G(X)$ and $1 + \overline{KO}_G(X)$ into Abelian groups. By invoking Brown's representation theorem we get the representing $G$-Hopf-spaces $BU_G^\otimes$ and $BO_G^\otimes$.

The map $\overline{K}_G(X) \to 1 + \overline{K}_G(X) : x \mapsto 1 + x$ is a bijection for every $G$-CW-complex $X$, and it follows that $BU_G^\otimes$ and $BU_G^\oplus$ are $G$-homotopy-equivalent $G$-Spaces. Similarly we see that $BO_G^\otimes$ and $BO_G^\oplus$ are equivalent $G$-spaces.

For later use we need the following:

**Proposition 1.7**

Let $X$ be a $G$-Space. If $E$ is a complex $G$-bundle over $X$, then there exists a $\mathbb{C}G$-module $M$ and a complex $G$-bundle $E^\perp$ such that $E \oplus E^\perp \cong M$ (where $M$ denotes the trivial $G$-bundle $M \times X \downarrow X$).

Similarly, if $F$ is a real $G$-bundle, then there is an $\mathbb{R}G$-module $N$ and a real $G$-bundle $F^\perp$ such that $F \oplus F^\perp \cong N$. 
Proof:
The complex case is (2.4) in [S68].
In the real case we do the following: $F \otimes_{\mathbb{R}} \mathbb{C}$ is a complex $G$-bundle, and we can thus find a complex $G$-bundle $F_1$, such that $(F \otimes_{\mathbb{R}} \mathbb{C}) \oplus F_1 \cong M$, where $M$ is a $\mathbb{C}G$-module. Now, $F$ is a direct summand of the underlying real $G$-bundle $r(F \otimes_{\mathbb{R}} \mathbb{C})$ of $F \otimes_{\mathbb{R}} \mathbb{C}$ with orthogonal complement $F_2$. By taking underlying real $G$-bundles, we obtain the relation
\[ F \oplus (r(F_1) \oplus F_2) \cong r(M). \]
Let $F^1 = r(F_1) \oplus F_2$, and $N = r(M)$.

QED

2. $G$-SO- and $G$-Spin-bundles

In this section we introduce $G$-SO-bundles and $G$-Spin-bundles, and we relate the classifying spaces of the functors $K_{SO_G}(-)$ and $K_{Spin_G}(-)$ to $BO_G$. We start by defining the $G$-spaces $BSO_G$ and $BSpin_G$ as the $G$-1-connected and $G$-2-connected cover of $BO_G$, respectively:

Recall that if $n > 1$ and $X$ is a $(n-1)$-connected space with $\pi_n(X)$ Abelian, then there is a map $k_n : X \to H(\pi_n(X), n)$, unique up to homotopy, such that $\pi_n(k_n)$ is the identity map on $\pi_n(X)$. Here $H(A,n)$ denotes the Eilenberg MacLane-space normally known as $K(A,n)$.

In the equivariant case we assume that $X$ is a $G$-$(n-1)$-connected $G$-CW-complex, i.e. for every subgroup $H$ of $G$ we have that the fixed point space $X^H$ is $(n-1)$-connected. We want to define a $G$-map $k_n : X \to H_G(\pi_n(X),n)$, where $H_G(A,n)$ is the equivariant Eilenberg-MacLane space classifying Bredon cohomology in dimension $n$ with coefficients in the $O_G$-group $A$, cf. [El], p. 277. $\pi_n(X)$ is the $O_G$-group sending the orbit $G/H$ to the Abelian group $\pi_n(X^H)$.

This map $k_n : X \to H_G(\pi_n(X),n)$ is defined as the element of $\left[ X, H_G(\pi_n(X),N) \right]^G$ corresponding to $k_n \in \left[ \Phi X, H(\pi_n(X),n) \right]_{O_G}$ under the bijection of [El], thm. 2. Here $k_n : \Phi X \to H(\pi_n(X),n)$ is given by
\[ k_n(G/H) = k_n : \Phi X(G/H) = X^H \to H(\pi_n(X^H),n) = H(\pi_n(X),n)(G/H). \]

Definition 2.1
Let
\[ w_i : BO_G \to H_G(\pi_i(BO_G),1) \]
be the map $k_i$ from above. Let $BSO_G$ denote the $G$-homotopy-fibre of $w_i$. ($k_i$ is well-defined, as $BO_G$ is $G$-connected, and $\pi_i(BO_G^H)$ is Abelian, cf. (1.5)).
Similarly, let 
\[ w_2 : BSO_G \to H^*_G(\tilde{\pi}_2(BSO_G) \otimes \mathbb{Z}/2, 2) \]
be the map \( r \circ k_2 \), where \( r : H^*_G(A, n) \to H^*_G(A \otimes \mathbb{Z}/2, n) \) is the mod 2 reduction map, and where \( k_2 : BSO_G \to H^*_G(\tilde{\pi}_2(BSO_G), 2) \) is defined as above. Let \( BSpin_G \)
denote the \( G \)-homotopy-fibre of \( w_2 \). (An argument using the \( G \)-fibration
\[ BSO_G \to BO_G \to H^*_G(\tilde{\pi}_3(BO_G), 1) \]
shows that \( BSO_G \) is \( G \)-1-connected, and \( k_2 : BSO_G \to H^*_G(\tilde{\pi}_2(BSO_G), 2) \) is thus well-defined.)

**Proposition 2.2**

Let \( U_1, U_2, ..., U_k \) be \( \mathbb{R}G \)-modules, \( V_1, V_2, ..., V_m \) be \( \mathbb{C}G \)-modules, and \( W_1, W_2, ..., W_n \) be \( \mathbb{H}G \)-modules, as in (1.5). Then
\[
BSO^G_G = \prod_{x=1}^{k} \prod_{y=1}^{m} \prod_{z=1}^{n} BSO \times BU \times BSp
\]
and
\[
BSpin^G_G = \prod_{x=1}^{k} \prod_{y=1}^{m} \prod_{z=1}^{n} BSpin \times BSpinU \times BSp
\]
where \( BSpinU \) is the homotopy-fibre of the composite map
\[ BU \xrightarrow{k} H(\pi_2(BU), 2) = H(\mathbb{Z}, 2) \xrightarrow{r} H(\mathbb{Z}/2, 2) \]
with \( r \) being the mod 2 reduction map.

**Proof:**

This follows immediately from (1.5) by taking the \( l \)-connected and 2-connected
covers of \( BO^G_G \). Recall that \( BSp \) is 2-connected, \( BU \) is 1-connected with \( \pi_1(BU) \cong \mathbb{Z} \), and that \( \pi_1(BO) \cong \mathbb{Z}/2 \) and \( \pi_2(BO) \cong \mathbb{Z} \).

QED

**Remark 2.3**

\( BSpinU \) is not the same space as \( BSpin^c \) of [St], p.292: We have that \( \pi_n(BSpinU) \cong \pi_n(BU) \) for \( n > 2 \), and especially \( \pi_6(BSpinU) \cong \mathbb{Z} \), while \( BSpin^c \) sits in the fibration sequence
\[ H(\mathbb{Z}, 2) \to BSpin^c \to BSO \]
and therefore \( \pi_6(BSpin^c) \cong \pi_6(BO) = 0 \).

From [L], p.257, we have the general definition of \( G \)-bundles, where \( A \) is the structure group. We explicify this definition in the cases where \( A = SO(n) \) or \( Spin(n) \):
**Definition 2.4**

A $G$-$SO$-bundle $E \hookrightarrow X$ of dimension $n$ is a $G$-map $p : E \to X$ between $G$-spaces such that

1) non-equivariantly, the map $p : E \to X$ is a $SO(n)$-bundle, and

2) for every $x \in X$ and $g \in G$ the restricted map $g |_{E_x} : E_x \to E_{gx}$ is a map of $G_x$-$SO$-modules.

If $E \hookrightarrow X$ and $F \hookrightarrow X$ are $G$-$SO$-bundles of the same dimension $n$, then a map $f : E \to F$ is a $G$-$SO$-bundle-map if $f$ is both a $G$-map and a $SO(n)$-bundle-map.

It is easily seen that the pull-back $f^* E$ along a $G$-map $f$ again is a $G$-$SO$-bundle. Furthermore, the pull-backs along $G$-homotopic maps of the same $G$-$SO$-bundle are equivalent $G$-$SO$-bundles. We define the direct sum $E \oplus F$ of two $G$-$SO$-bundles $E \hookrightarrow X$ and $F \hookrightarrow X$ as $E \oplus F = \Delta^* (E \times F)$, where $\Delta : X \to X \times X$ is the diagonal map.

Finally, we get the Grothendieck-group $KSO_G(X)$ of isomorphism-classes of $G$-$SO$-bundles over the $G$-space $X$, and we define $\overline{KSO}_G(X)$ as the subgroup of $KSO_G(X)$ generated by differences of bundles $E \to F$ satisfying $\forall x \in X : E_x \cong F_x$ as $G_x$-$SO$-modules.

**Definition 2.5**

A $G$-Spin-bundle $E \hookrightarrow X$ of dimension $n$ is two $G$-spaces $E$ and $X$ and a $G$-map $p : E \to X$ such that

1) $p : E \to X$ is non-equivariantly a $Spin(n)$-bundle, and

2) for every $x \in X$ and $g \in G$ the restricted map $g |_{E_x} : E_x \to E_{gx}$ is a morphism of $G$-Spin-modules.

As with $G$-$SO$-bundles we get a Grothendieck-group $KSpin_G(X)$ and a reduced version $\overline{KSpin}_G(X)$.

**Theorem 2.6**

The classifying spaces of the functors $KSO_G(\_)$ and $KSpin_G(\_)$ are $BSO_G$ and $BSpin_G$, respectively.

**Proof:**

We denote momentarily the classifying spaces for the functors $KSO_G(\_)$ and $KSpin_G(\_)$ by $B_1$ and $B_2$. We construct $G$-maps $\phi : B_1 \to BSO_G$ and $\psi : B_2 \to BSpin_G$ and show that they are $G$-homotopy-equivalences.

The spaces $B_1$ and $B_2$ are $G$-connected, as for every subgroup $H$ of $G$, we have that

$$\pi_0(B_1^H) = \overline{KSO}_G(S^0 \wedge (G/H)_+^H) = 0$$
We have a 'forgetful' map $\varphi : G KSO(X) \to G KO(X)$ for every $G$-connected $G$-CW-complex $X$, defined by sending a $G$-$SO$-bundle to its underlying orthogonal $G$-bundle. As $\varphi$ is a natural transformation between functors, we get a $G$-map $\varphi' : B_1 \to BO_G$.

Let $E - F \in G KSO_G(S^1)$, where $E$ and $F$ are $G$-$SO$-bundles. We decompose $E$ (and $F$) according to [K], (4.1): Using the notation of (1.5), we can find real bundles $\eta_1, \eta_2, \ldots, \eta_k$, complex bundles $\zeta_1, \zeta_2, \ldots, \zeta_m$, and symplectic bundles $\xi_1, \xi_2, \ldots, \xi_n$, such that

$$E = U_1 \oplus \eta_1 \oplus \ldots \oplus V_1 \oplus \zeta_1 \oplus \ldots \oplus W_n \oplus \xi_n.$$

All the $\eta_i$'s are $SO$-bundles, as the $SO$-action on $E$ in $\eta_i = \text{Hom}_{BG}(U, E)$ gives a $SO$-action on $\eta_i$. Furthermore, our decomposition of $E$ above is easily seen to be a decomposition of $G$-$SO$-bundles. Now, as $G KSO(S^1) = K(S^1) = G KSp(S^1) = 0$, all $SO$, $U$- and $Sp$-bundles over $S^1$ are trivial. Especially, the $\eta_i$'s, the $\zeta_j$'s and the $\xi_k$'s are trivial bundles, and $E$ becomes a trivial $G$-bundle. We see that $G KSO_G(S^1) = 0$, and as $G KSO_G(S^1 \wedge (G / H)) \cong G KSO_H(S^1)$, we conclude that $B_1$ is $G$-1-connected.

The map $w_1 \circ \bar{\phi} : B_1 \to H_0 G(\pi_2 BO_G, 1)$ is null-homotopic, as $[B_1, H_0 G(\pi_2 BO_G, 1)]^G = H_0 G(B_1 ; \pi_2 BO_G)$ is zero: $B_1$ is $G$-1-connected, and [Br], (11.7.1) shows that $B_1$ is $G$-homotopy-equivalent to a $G$-complex with no cells in dimensions less that 2. The definition of $G$-cohomology, [Br], (1.6.4), implies that $H_0 G(B_1 ; \pi_2 BO_G)$ vanishes.

We get a lift $\phi : B_1 \to BSO_G$ of $\bar{\phi}$. We show that for every finite, $G$-connected $G$-CW-complex $X$ the induced map $\phi : G KSO_G(X) \to [X, G BSO_G]^G$ is an isomorphism. By using the equivariant Whitehead theorem and the fact that $G KSO_G(S^n \wedge (G / H)) = G KSO_H(S^n)$, it suffices to consider the case where $X = S^n$, $n \geq 1$. For $n = 1$, both $G KSO_G(S^1)$ and $[S^1, BSO_G]^G$ are zero.

Let $E$ and $F$ be $G$-bundles over $S^n$, $n > 1$, and let $E - F$ represent an element of $[S^n, BSO_G]^G = G KO_G(S^n)$. By using the decomposition (2.7), we get orthogonal bundles $\eta_i$ over $S^n$. As $KO(S^n) = KSO(S^n)$, the $\eta_i$'s are actually $SO$-bundles, and $E$ becomes a $G$-$SO$-bundle (the complex and symplectic parts of $E$ give no problem here). Thus, we see that $\phi$ is surjective.

To show that $\phi$ is injective, we show that the composite $\bar{\phi}$ is injective. So, let $E - F \in \text{Ker}(\bar{\phi})$. Decompose $E$ and $F$ as above and note that we have $O$-isomorphisms between $\eta_i$, $\pi_i$, $U$-isomorphisms between $\zeta_j$, and $\pi_j$, and $Sp$-isomorphisms between $\xi_k$ and $\pi_k$. But on $S^0$ there is no difference between $O$-isomorphisms and $SO$-isomorphisms of vector-bundles, as $KO(S^0) \cong G KSO(S^0)$,
and as $U$- and $Sp$-isomorphisms are $SO$-isomorphisms, we get $SO$-isomorphisms on all the components in the decompositions of $E$ and $F$. These are assembled to show that $E \equiv F$ as $G$-$SO$-bundles, and we see that $E - F = 0$, and $\Phi$ is injective. This shows that $B_1 = BSO_G$.

The part of the theorem concerning $\overline{KSpin}_G(-)$ and $BSpin_G$ is proved in the same way: The map $\overline{\psi}: B_2 \to BO_G$ is defined as the 'forgetful' map sending a $G$-$Spin$-bundle to its underlying orthogonal bundle. By using methods as above, we see that $w_1 \circ \overline{\psi}$ and $w_2 \circ \overline{\psi}$ are null-homotopic, and we get a $G$-map $\psi: B_2 \to BSpin_G$ – one of the main points is that if $E \downarrow X$ is a complex bundle, then the obstruction to $E$ being a $Spin$-bundle is $w_2(E) \in H^2(X; \mathbb{Z}/2)$. But $w_2(E)$ is the image of $c_1(E)$ under the reduction map $H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}/2)$. (It is this fact that makes the use of the space $BSpin U$ necessary). By showing that the decomposition (2.7) respects $Spin$-structures, we see as before that $\psi$ is a $G$-homotopy-equivalence.

QED

We remark that the $G$-spaces $BSO_G$ and $BSpin_G$ are $G$-Hopf-spaces, cf. [Br], §11.4: The maps $w_1$ and $w_2$ are seen to be Hopf-maps by considering the functionality of the Elmendorfer construction – the map $k_n: X \to H(\pi_n(X), n)$ will in general be a Hopf-map when $X$ is a Hopf-space. $BSO_G$ and $BSpin_G$ with this Hopf-structure is denoted by $BSO_G^{\oplus}$ and $BSpin_G^{\oplus}$.

The tensorproduct of $G$-$SO$- and $G$-$Spin$-bundles gives the Hopf-spaces $BSO_G^{\oplus}$ and $BSpin_G^{\oplus}$ representing the functors $1 + \overline{KSOG}(-)$ and $1 + \overline{KSpin}_G(-)$. As it is the case with $BO_G$, we have that $BSO_G^{\oplus}$ and $BSO_G^{\otimes}$, and that $BSpin_G^{\oplus}$ and $BSpin_G^{\otimes}$ are equivalent $G$-spaces, but the Hopf-space-structures will in general be different.

For later use we describe the rational type of $BSO_G$:

**Lemma 2.8**

Let $q$ be a prime not dividing the order of the group $G$. Let $X$ be a $G$-space, and let $Y$ be a $q$-local infinite $G$-loop space. Then the $q$-local map

$Fix: [X, Y]_q^G \to [\Phi_\mathcal{X}, \Phi_\mathcal{Y}]_{c_q(q)}$

sending the $G$-map $f: X \to Y$ to the $\mathcal{O}_G$-map

$Fix(f): G/H \mapsto (f^H: X^H \to Y^H)$

is a bijection.

**Proof:**

This is essentially [LMS], (V.6.8) and (V.6.9): If $(|G|, q) = 1$, then

$[X, Y]_q^G \cong \prod_{(H)} [X^H, Y^H]_{q, INV}^{\mathcal{O}_G}$
where the superscript 'INV' indicates that we are considering homotopy classes of 'invariant maps', [LMS] (V.6.5). But such an invariant homotopy class corresponds to a $O_G$-homotopy class of $O_G$-maps $\Phi X \to \Phi Y$.

QED

**Proposition 2.9**

Let $BSO_G^Q$ be the representing space of the functor $KSO_G(-) \otimes Q$. Then

$$BSO_G^Q \cong \prod_{n=2} H_G(\pi_n(BSO_G) \otimes Q, n)$$

**Proof:**

From (2.8) we have

$$\overline{KSO_G}(X) \otimes Q \cong [X, BSO_G^Q]^G \cong [\Phi X, \Phi BSO_G^Q]_G$$

For a subgroup $H$ of $G$ we have that

$$BSO_G^Q H \cong \prod_{i=1}^k BSOQ \times \prod_{j=1}^m BUQ \times \prod_{z=1}^n BSpQ$$

as it follows from (2.2), and where $BSOQ$, $BUQ$ and $BSpQ$ are the rational types of $BSO$, $BU$ and $BSp$, respectively.

It is well-known that

$$BSOQ \cong \prod_{n=2} H(\pi_n(BSO) \otimes Q, n)$$

and similarly for $BU$ and $BSp$, and we see that

$$BSO_G^Q H \cong \prod_{n=2} H_G(\pi_n(BSO_G^H) \otimes Q, n)$$

By applying [El], thm. 2, we get the result.

QED

Of course, similar results holds for $BO_G^Q$, $BU_G^Q$, $BSp_G^Q$ and $BSpin_G^Q$.

### 3. The structure of $BSO_G$

In this section we study the structure of the space $BSO_G$ via the $\lambda$-ring-structure on the functor $KSO_G(-)$. The aim is to generalize results of Atiyah-Tall and Atiyah-Segal.

In the following we assume that $G$ is a group of odd order. This implies that the numbers $k$ and $n$ of (1.5) are 1 and 0, respectively. Furthermore, $\pi_n(BO_G)$ is the constant coefficient system $\mathbb{Z}/2$.

We start by showing an equivariant analogue of the splitting principle in Bredon cohomology, cf. [Hu], (16.5.2).
**Lemma 3.1**

Let $E \downarrow X$ be a $G$-bundle. Then there is a $G$-space $Q(E)$ and a $G$-map $q : Q(E) \rightarrow X$ such that $q^*(E)$ splits as a sum of $G$-line-bundles and the map

$$q^* : H_G^*(X; \mathbb{Z}_G(BO_G)) \rightarrow H_G^*(Q(E); \mathbb{Z}_G(BO_G))$$

is a monomorphism.

**Proof:**

As in the non-equivariant case, [Hu] (16.5.2), we construct $Q(E)$ inductively by going from $X$ to $P(E)$ — the projective bundle of $E$. We see that the bundle $p^*(E)$ over $P(E)$ splits as a sum of a canonical line-bundle and another bundle of lower dimension than $E$, and we repeat this procedure on the latter bundle. (Here $p : P(E) \rightarrow X$ is the projection on the base space).

The injectivity of the map in Bredon-cohomology is also shown stepwise. It suffices to show that the map

$$p^* : H_G^*(X; \mathbb{Z}_G(BO_G)) \rightarrow H_G^*(P(E); \mathbb{Z}_G(BO_G))$$

is injective.

As the order of the group $G$ is odd, and the coefficient system $\mathbb{Z}_G(BO_G)$ is a $\mathbb{Z}^{(2)}$-module, we get from [LMS], (V.6.8) and (V.6.9), that there is a natural isomorphism

$$\Phi : H_G^*(Z; \mathbb{Z}_G(BO_G)) \rightarrow \bigoplus_{\langle H \rangle} H^*(Z^n; \mathbb{Z}/2)$$

Here the sum is over all conjugacy classes of subgroups of $G$.

Using (3.2), we reduce the problem to show that

$$(q^H)^* : H^*(X^n; \mathbb{Z}/2) \rightarrow H^*(P(E)^n; \mathbb{Z}/2)$$

is injective for every subgroup $H$ of $G$. But as $G$ is of odd order $P(E)^n$ equals the projective bundle of the real bundle $E^n |_{X^n} \rightarrow X^n$, and we now use the non-equivariant splitting principle of [Hu], (16.5.2).

QED

If $E \downarrow X$ is a real $G$-bundle, we define $w_i(E) \in H_G^i(X; \mathbb{Z}_G(BO_G))$ as $w_i(E-V)$, where $V$ is the trivial bundle having $V = E$ as fibre. If $w_i(E) = 0$, we say that $E$ is $G$-orientable.

**Lemma 3.3**

Let $E$ and $F$ be $G$-line-bundles over the $G$-connected $G$-space $X$. Then

$$w_i(E \oplus F) = w_i(E) + w_i(F).$$

**Proof:**

Let $L_G(X)$ be the semi-group of $G$-line-bundles over $X$ with $\otimes$ as the composition. $L_G(\cdot)$ is clearly a representable functor. Denote the classifying space by $BL_G$. Since $L_G(X)$ has a natural multiplication for all $X$, we see that $BL_G$ is a $G$-Hopf-space.
We now get the following homotopy commutative diagram:

\[
\begin{array}{ccc}
 BL_G & \xrightarrow{k_i} & H_G(\eta_i(BO),1) \\
 \downarrow & & \downarrow j \\
 BO_G & \xrightarrow{n_j} & H_G(\eta_i(BO),1)
\end{array}
\]

where the map \( i \) is induced by the map

\[
L_G(X) \rightarrow KO_G(X): E \mapsto E - E,
\]

and \( j \) comes from the \( O_G \)-group-homomorphism \( \eta_i(j): \eta_i(BO) \rightarrow \eta_i(BO) \).

All these maps except possibly \( i \) are Hopf-maps. The commutativity of the diagram now gives the result.

QED

**Corollary 3.4**

Assume \( X \) is a \( G \)-connected \( G \)-space. Then \( KO_G(X) \) is stable under the multiplication induced by \( \otimes \).

**Proof:**

It suffices to show that if \( E \) and \( F \) are \( G \)-orientable then \( E \otimes F \) is \( G \)-orientable, too. By using the splitting principle (3.1), we reduce to the case where \( E \) and \( F \) are line-bundles, and (3.3) gives the result.

QED

We recall that \( KO_G(X) \) is a \( \lambda \)-ring: If \( E \) is a \( G \)-bundle over \( X \) and \( n \) a non-negative integer, then \( \lambda^n E \) is the real \( G \)-bundle \( \Lambda^n E \), where the \( G \)-action is given by

\[
g(e_1 \wedge e_2 \wedge ... \wedge e_n) = (ge_1) \wedge (ge_2) \wedge ... \wedge (ge_n).
\]

**Proposition 3.5**

Let \( X \) be a finite \( G \)-connected \( G \)-CW-complex. Then \( KO_G(X) \) is a special, finite-dimensional \( \lambda \)-ring.

**Proof:**

\( KO_G(X) \) is finite-dimensional, as every real \( G \)-bundle is finite-dimensional: Let \( E \) be a \( G \)-bundle over \( X \), where \( n \) the dimension of a fibre of \( E \). Then \( \Lambda^m E = 0 \) for \( m > n \).

That \( KO_G(X) \) is a special \( \lambda \)-ring follows from the splitting principle in \( KG \)-theory; see [tD], p.32.

QED

**Corollary 3.6**

\( KSO_G(X) \) is a special \( \lambda \)-ring. \( KSO_G(X) \) is a \( \lambda \)-ideal in \( KSO_G(X) \).
Proof:
We must show that if $E$ is a $G$-oriented $G$-bundle, then $\Lambda^n E$ is $G$-oriented for all integers $n$. Using the splitting principle (3.1), we may assume that $E$ is a sum of linebundles, $E = F_1 \oplus F_2 \oplus \ldots \oplus F_m$. We have the isomorphism
\[
\Lambda^n(F_1 \oplus F_2 \oplus \ldots \oplus F_m) = \bigoplus(F_{i(1)} \oplus \ldots \oplus F_{i(n)}),
\]
where the sum is over all sequences $i(1) < i(2) < \ldots < i(n)$ of integers, cf. [Hu], (5.6.10). By using (3.3) we see that $w_i(\Lambda^n E)$ equals $\binom{m}{n} w_i(E) = 0$.
QED

Proposition 3.7
For $X$ $G$-connected, the $\gamma$-ring $KSO_G(X)$ is an oriented $\gamma$-ring.

Proof:
According to [AT], p.285 it suffices to show that for every $x \in KSO_G(X)$ there exist $G$-bundles $E$ and $F$ over $X$ such that $x = E - F$, and, if $n$ denotes the dimension of $E$ and $F$, then the linebundles $\Lambda^n E$ and $\Lambda^n F$ are the trivial one-dimensional $G$-bundle $V \times X \downarrow X$.

Write $x$ as $E - V$, where $E$ is a $G$-bundle and $V$ is the trivial bundle $V \times X \downarrow X$ for some $G$-module $V$, as in (1.7). Discarding the $G$-actions for a moment, we see that
\[
0 = w_i(x) = w_i(E) - w_i(V) = w_i(E)
\]
and thus both $\Lambda^n E$ and $\Lambda^n F$ are trivial line-bundles, as $KSO(X)$ is an oriented $\lambda$-ring. We decompose $\Lambda^n E$ as in (1.5). As $\Lambda^n E$ is one-dimensional, this decomposition must be of the form $\Lambda^n E \cong \mathbb{R} \otimes \eta$, as $\mathbb{R}$, the trivial one-dimensional representation, is the only 1-dimensional representation of $G$. If we ignore the $G$-action, $\mathbb{R}$ gives the trivial line-bundle, and $\Lambda^n E \cong \eta$ is a trivial bundle. Thus, both $\Lambda^n E$ and $\Lambda^n V$ are isomorphic to $\mathbb{R}$.
QED

From now on we assume that $p$ is an odd prime, and that $G$ is a $p$-group.

Proposition 3.8
Let $X$ a $G$-connected $G$-CW-complex. Then $KSO_G(X) \otimes \mathbb{Z}_p$ is a $p$-adic $\gamma$-ring.

Proof:
As $X$ is $G$-connected, the natural inclusion $KSO_G(X) \otimes \mathbb{Z}_p \rightarrow KO_G(X) \otimes \mathbb{Z}_p$ is a monomorphism preserving the $\gamma$-ring-structure. [ID], (3.8.6) now gives the result.
QED
Theorem 3.9
There is a splitting of $G$-Hopf-spaces:
\[
(BSO_G)_p^\wedge \cong B_0^\otimes \times B_1^\otimes \times \ldots \times B_{m-1}^\otimes , \quad m = \frac{p-1}{2}
\]

Proof:
[AT], lemma 2.2, p.279 shows that, as $\overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ is a $p$-adic $\gamma$-ring, the domain of the Adams operations $\psi^k : \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p \to \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ in the variable $k$ extends by continuity to operations
\[
\psi^a : \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p \to \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p ,
\]
where $a \in \hat{\mathbb{Z}}_p$.

Letting $\alpha$ be a generator of the finite factor $\mathbb{Z}/(p-1)$ of the splitting
\[
(\hat{\mathbb{Z}}_p)^* \cong \mathbb{Z}/(p-1) \times \hat{\mathbb{Z}}_p
\]
we have from [AT], p.284, that $\overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ splits canonically into eigenspaces for the operator $\psi^a$, the eigenvalues being $\alpha', i = 0, 1, \ldots, p-2$.

As this splitting is canonical in the space $X$, we get a corresponding splitting of the classifying space $(BSO_G)_p^\wedge$ into $p-1$ components.

Half of these components vanish: Let $i$ be one of the odd numbers $1, 3, \ldots, p-2$, and let $x \in \overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ be an eigenvector for $\psi^a$ with eigenvalue $\alpha'$. As $\alpha^{(p-1)/2} = -1$, [AT], (5.2), p.264, shows that
\[
\psi^{-1}(x) = \psi^{\alpha^{(p-1)/2}}(x) = \alpha'^{(p-1)/2} x = -x
\]
But as $\overline{KSO}_G(X) \otimes \hat{\mathbb{Z}}_p$ is an oriented $\gamma$-ring, (3.7), $\psi^{-1}$ acts as the identity operator, see [AT], p.285. Thus, $x$ must be 0, and these all of 'odd' components vanish.

QED

Theorem 3.10
Let $p$ be an odd prime. Then there is a splitting of $G$-Hopf-spaces
\[
(BSO_G)_p^\wedge \cong B_0^\otimes \times B_1^\otimes \times \ldots \times B_{m-1}^\otimes , \quad m = \frac{p-1}{2}
\]

Proof:
The proof is the same as that of (3.9) - the Adams operation $\psi^a$ acts on $1 + \overline{KSO}_G(X)_p^\wedge$ by $\psi^a(x) = x^{a'}$, where $x$ is an element of the $i$'th eigenspace.

QED

Theorem 3.11
Let $p$ be an odd prime and let $k$ be an integer such that $k + p^2\mathbb{Z}$ generates the group of units in the ring $\mathbb{Z}/p^2\mathbb{Z}$. Then the cannibalistic class $\rho^k$ induces an $G$-homotopy-equivalence of $G$-Hopf-spaces
\[
\rho^k : B_0^\otimes \to B_0^\otimes
\]
Oriented, equivariant $K$-theory and the Sullivan splittings
Kenneth Hansen

Proof: [AT] (II 4.4).

QED

We obtain from [AS] thm.2 :

**Theorem 3.12**
Let $p$ be a prime. Then there is a $G$-homotopy equivalence of $G$-Hopf-spaces

$$\delta : (BSO_G^\oplus)_p^\wedge \rightarrow (BSO_G^\oplus)_p^\wedge$$

A variant of $\delta$ is $\bar{\delta}$, which is the map $\rho^k$ at the first component $B_0$ and $\delta$ at the rest of the components. As above we see that

$$(3.13) \quad \bar{\delta} : (BSO_G^\oplus)_p^\wedge \rightarrow (BSO_G^\oplus)_p^\wedge$$

is an equivalence of $G$-Hopf-spaces.

**Remark 3.14**
Actually, the results of [AT] and [AS] cannot be used directly in (3.9)-(3.13): In [AT] and [AS] it is assumed that we have a $\lambda$-ring $R$ with an augmentation $\varepsilon : R \rightarrow \mathbb{Z}$, and then the results of [AT] holds for the augmentation ideal $I$.

We are in a more general situation, in that we have the $\lambda$-ring $KSO_G(X)$ and the $\lambda$-homomorphism $\varepsilon : KSO_G(X) \rightarrow RO(G)$ sending a $G$-bundle $E$ to the representation $E$. The kernel of $\varepsilon$ is $KSO_G(X)$. It is possible to generalize the results of [AT] and [AS] to this case without any serious difficulties.

**Counterexample 3.15**
The crucial step in getting (3.9)-(3.12) from [AT] and [AS] is (3.8). When $G$ is not a $p$-group, or when we do not localize at the order of the group, (3.8) does not hold. We give a simple counterexample:

If (3.8) did hold, then we would have, as in (1.5.6) in [AT], that the Adams' operation $\psi^k : KSO_G(X) \rightarrow \overline{KSO_G(X)}$ would be $p$-adically continuous in the variable $k$.

Let $G = \mathbb{Z}/3$ be the cyclic group of order 3, and let $p$ be the prime 5. Then $KSO_G(S^{2n})$ is isomorphic to $RO(G)$ and is a free $\mathbb{Z}$-module of rank 2 with generators $1, V$ corresponding to the two irreducible $\mathbb{R}G$-modules of dimension 1 and 2, respectively. $\psi^k$ maps $a1+bV$ to $k^{2n}(a1+bV)$ if $(k,3)=1$ and to $k^{2n}(a+2b)$ if $3|k$. 
If $\psi^k$ was 5-adically continuous in $k$, then for every $x \in KSO_G(S^{4n})$ and integer $m$ we could find an integer $r$, such that for $S' | s$ and integer $k$, we would have

$$\psi^{k+s}(x) - \psi^k(x) \in 5^m \cdot KSO_G(S^{4n})$$

But if $3 | (k + s)$ and $(3, k) = 1$ and $x = a1 + bV$, then

$$\psi^{k+s}(x) - \psi^k(x) = (((k + s)^2 - k^2)a + 2k^2b)1 + k^2V$$

which definitely not is contained even in $5 \cdot KSO_G(S^{4n})$.

4. $SF_G$ and the Adams' conjecture

We now proceed to study the $G$-space $SF_G$. Important ingredients in this analysis is the equivariant Adams' conjecture, due to McClure, cf. [MC], and the results of §3. Our standing assumption is that $p$ is an odd prime, $G$ is a $p$-group, and that all spaces are $p$-local.

**Definition 4.1**

Let $Q_G S^0$ be the $G$-loop-space $\lim_{\to} \Omega^V S'$, where the limit is over all $G$-modules in a fixed, complete $G$-universe $\mathcal{U}$, cf. [LMS] p. 11. $Q_G S^0$ is a 'G-ring-space', where the additive structure comes from the 'loop-sum' $*: \Omega^V S' \to \Omega^V S'$, which exists for every $G$-module $V$, and where the multiplication is composition of maps. We let the identity map be the basepoint of $Q_G S^0$.

Let $SF_G$ be the $G$-connected cover of $Q_G S^0$. $SF_G$ inherits a (multiplicative) $G$-Hopf-space structure from $Q_G S^0$.

Certain facts about $Q_G S^0$ are well-known - we recall from [S70], p.62, that

$$(Q_G S^0)^G \cong \prod_{(H)} Q(BW_H),$$

where the product is over all conjugacy classes $(H)$ of subgroups of $G$. $W_H$ is the Weyl-group $N_G(H)/H$. By taking connected covers we see that

$$(SF_G)^G \cong \prod_{(H)} Q_0(BW_H),$$

where $Q_0(BW_H)$ is the basepoint component of $Q(BW_H)$.

**Definition 4.4**

Let $X$ be a finite $G$-$CW$-complex. The $G$-fibration $\xi: E \to X$ is a spherical $G$-fibration or a $G$-sphere-bundle, if

1) for every $x \in X$ there is a $G_x$-representation $V$ such that the fibre $E_x$ is $G_x$-homotopy-equivalent to $S^V$, and
2) the map $X \rightarrow E$ given by $x \mapsto (the$ basepoint of $E_x)$ is a $G$-cofibration.

(This is the definition of [MC], p.230-231).

Fibre-wise smash-products makes the set of $G$-sphere-bundles over $X$ into a semigroup, and the corresponding Grothendieck group is denoted $KF_G(X)$. The subgroup $\overline{KF}_G(X)$ is defined as follows

\[(4.5) \quad E - F \in \overline{KF}_G(X) \quad \iff \quad \forall x \in X : E_x \simeq F_x \ as \ G_x - spaces.\]

The functors $KF_G(-)$ and $\overline{KF}_G(-)$ are easily seen to be representable functors. We denote the classifying space of $\overline{KF}_G(-)$ by $BF_G$.

It follows from [W] that

\[(4.6) \quad \pi_0(BF_G) = 0 \quad and \quad \pi_1(BF_G) \cong \mathcal{A}(G)^\ast,\]

where the $\mathcal{O}_G$-group $\mathcal{A}(G)^\ast$ is given by $\mathcal{A}(G)^\ast(G/H) = A(H)^\ast$ – the unit group of the Burnside ring $A(H)$. Furthermore, we see that $BF_G$ is the classifying $G$-space of the $G$-monoid $F_G$ – the subspace of $Q_G S^0$ consisting of $G$-homotopy-equivalences with the monoid structure coming from composition of maps.

Let $BSF_G$ be the 1-connected cover of $BF_G$. It follows that $BSF_G$ is the classifying space of the monoid $SF_G$, and thus

\[(4.7) \quad \Omega BGSF_G \cong SF_G.\]

Define the natural transformation $J_G : KO_G(X) \rightarrow KF_G(X)$ by sending the real $G$-bundle $E \downarrow X$ to its fibrewise one-point compactification $S^E \downarrow X$. It is immediately seen that $J_G$ restricts to a natural transformation $KO_G(X) \rightarrow \overline{KF}_G(X)$, and thus produces a $G$-Hopf-map $J_G : BO_G \rightarrow BF_G$. Furthermore, by killing off $\pi_1$, we get a lift of $J_G : BSO_G \rightarrow BSF_G$.


**Proposition 4.8**

The natural map $\theta : SF/ SO_G \rightarrow F/O_G$ is a $G$-homotopy equivalence if $G$ is of odd order or if we localize at an odd prime $p$.

**Proof:**

We have the $G$-homotopy commutative diagram:
Oriented, equivariant $K$-theory and the Sullivan splittings
Kenneth Hansen

\[
\begin{align*}
SF / SO_G & \longrightarrow BSO_G & J_G & \rightarrow BSG_G \\
0 \downarrow & & \downarrow & \\
F / O_G & \longrightarrow BO_G & J_G & \rightarrow BF_G \\
\downarrow & & \downarrow & \\
H_G(\pi_1(BO_G), 1) & \longrightarrow H_G(\pi_1(BF_G), 1)
\end{align*}
\]

Let $H$ be a subgroup of $G$. $\pi_1(BO_G^H) \cong RO(H) / R(H)$ and $\pi_1(BF_G^H) = A(H)^\times$ are both 2-torsion groups, and $0$ is thus an equivalence away from 2.

If $G$ is of odd order, then both $RO(H) / R(H)$ and $A(H)^\times$ are isomorphic to $\mathbb{Z}/2$. Furthermore, the non-zero element in $K_G(S^1 \wedge (G / H)^\times) \cong KO_H(S^1)$ is represented by the reduced Möbius-bundle with trivial $G$-action and, as in the non-equivariant case, is mapped by $J_G$ to the non-trivial element in $KF_G(S^1 \wedge (G / H)^\times)$. Thus $\psi$ is a $G$-homotopy equivalence and the result follows.

QED

The Adams conjecture relates $J_G$ to the Adams-operations in $K$-theory. The non-equivariant version states:

Let $k$ be an integer, $x \in KO(X)$. Then there exist an integer $n$, such that

\[k^n J(\psi^k x - x) = 0.\]

By localizing at a prime $p$, satisfying $(p, k) = 1$, we get rid of the factor $k$. Various attempts have been made to generalize the Adams conjecture to the equivariant case. In [FHM], theorem 0.4, it is shown that $k^n s J(\psi^i x - x) = 0$, where $(k, |G|) = 1$, and $s$ is the minimal integer, such that $k^s \equiv \pm 1 \pmod{|G|}$. The extra factor $s$ is necessary – it insures that the 'fibres' of the virtual $G$-bundles $\psi^k x$ and $x$ are the same element in $R(G_a)$ for every $a \in X$.

McClure has another variation, cf. [MC] (5.1). This uses a variant of the functor $KF_G(X)$:

Let $p$ be a prime. Define the equivalence relation $\sim$ of stable $p$-equivalence on $KF_G(X)_{(p)}$ as follows: The $G$-sphere-bundles $E$ and $F$ are stably $p$-equivalent if there exists a real $G$-representation $V$ and $G$-fiber maps

\[f_1 : S^v E \to S^v F \quad \text{and} \quad f_2 : S^v F \to S^v E\]

such that $f_1$ and $f_2$ have degrees prime to $p$ on all fixed sets of each fibre.

Denote the set of stably $p$-equivalence classes in $KF_G(X)_{(p)}$ by $KF_G^{(p)}(X)$, and denote the reduced version by $\overline{KF_G}^{(p)}(X)$. 
The relation between $KF_G^*(X)_{(p)}$ and $KF_G^{(p)}(X)$ is as follows, cf. [MC], (1.3):

Let $X$ be a $G$-connected, finite $G$-CW-complex. Then there is a natural, short exact sequence

$$0 \rightarrow jO(G) \overset{\alpha}{\longrightarrow} KF_G^*(X)_{(p)} \longrightarrow KF_G^{(p)}(X) \rightarrow 0$$

where $jO(G) = RO_0(G)/RO_1(G)$ ([tD] p.229), and $\alpha$ is the composite

$$jO(G) \rightarrow \text{Im}(J : KO_G(*)_{(p)} \rightarrow KF_G(*)_{(p)}) \rightarrow KF_G(X)_{(p)}$$

**Lemma 4.9**

For $X G$-connected we have $KF_G^*(X)_{(p)} \cong KF_G^{(p)}(X)$.

**Proof:**

We have the exact commutative diagram

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & KF_G^*(X)_{(p)} & KF_G^{(p)}(X) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & jO(G)_{(p)} & KF_G^*(X)_{(p)} & KF_G^{(p)}(X) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & jO(G)_{(p)} & KF_G(*)_{(p)} & KF_G^{(p)}(*) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0
\end{array}
$$

QED

The equivariant Adams' conjecture [MC], (5.1) is now

**Theorem 4.10**

Let $p$ be an odd prime and let $k$ be an integer prime to $p$ and $|G|$. Then the composite

$$(BSO_G)_{(p)} \overset{\psi^{1-1}}{\longrightarrow} (BSO_G)_{(p)} \overset{J}{\longrightarrow} (BSF_G)_{(p)}$$

is null-homotopic.

Actually, this is not McClures formulation of the Adams conjecture, but upon using reduced $KO_G$- and $KF_G$-groups, and by using (4.9), we get the result above. The reason why this formulation doesn't involve extra factors is that we work in
reduced $KO_G$- and $KF_G$-theory. This means that the condition that $\xi$ and $\xi^k$ have the same fibres over $x$ in $R(G_x)$ for $x \in X$, is automatically fulfilled.

**Corollary 4.11**

There is a map $\alpha_k : (BSO_G)_G(p) \to (F/O_G)_G(p)$ such that

$$(F/O_G)_G(p) \xymatrix{ \ar[r] & (BSO_G)_G(p) \ar[r]^-j & (BSF_G)_G(p) \ar[l]^-{\alpha_k} \ar@{<=>}[u]_{\psi^k - 1} }$$

commutes up to homotopy.

**Definition 4.12**

Let $G$ be a group of odd order, and let $p$ be an odd prime. Let $k$ be an integer, such that $2^k p^2 \mathbb{Z}$ generates the unit group $(\mathbb{Z}/p^2 \mathbb{Z})^\times$. Define the $G$-Spaces $J_G^\oplus$ and $J_G^\otimes$ as the homotopy fibres of the maps $\psi^k - 1 : BSO_G^\oplus \to BSO_G^\oplus$ and $\psi^k / 1 : BSO_G^\otimes \to BSO_G^\otimes$. As both $\psi^k - 1$ and $\psi^k / 1$ are Hopf-maps, $J_G^\oplus$ and $J_G^\otimes$ becomes $G$-Hopf-spaces. $J_G^\oplus$ and $J_G^\otimes$ are equivalent $G$-Spaces, but the Hopf-structures will in general be different.

**Remark 4.13**

In [FHM], (0.5) it is shown that $J_G$ is the $G$-connected cover of equivariant, orthogonal, algebraic K-theory, $KO(\mathbb{F}_p, G)$, provided that $k$ is a prime power.

5. The $e$-invariant and the Sullivan splittings

We now generalize the splittings $F/O \simeq BSO \times \text{Cok} J$ and $SF \simeq J \times \text{Cok} J$ of Sullivan to the equivariant case. We already have one of the maps needed to prove this, namely $\alpha_k$, and we now define the other – the $e$-invariant.

As usual, $p$ is an odd prime, $G$ is a $p$-group, all spaces are $p$-local, and $k$ is an integer such that $2^k p^2 \mathbb{Z}$ generates the unit group $(\mathbb{Z}/p^2 \mathbb{Z})^\times$.

The main reason for studying $G$-Spin-bundles is that, as in the non-equivariant case, a $G$-Spin$(8n)$-bundle has a Thom-class in $KO_G$-theory. Recall from [A], (6.1):
**Theorem 5.1**

Let $\Pi$ be a compact Lie group, $V$ a $\Pi$-$\text{Spin}$-module of dimension $8n$, and $X$ a compact $G$-Space. Then there is an element $u \in KO_G(X)$, defined by using the Dirac operator on $V$, such that multiplication with $u$ induces an isomorphism

$$KO_G(X) \to KO_G(X \times V)$$

**Theorem 5.2**

Let $G$ be a finite group, $E \downarrow X$ a $G$-$\text{Spin}(8n)$-bundle over the compact $G$-connected $G$-$\text{CW}$-space $X$. Then there is an isomorphism

$$\Phi_E : KO_G(X) \to KO_G(T(E))$$

where $T(E)$ is the Thom-complex of $E$.

**Proof:**

Let $R \downarrow X$ be the principal $G$-$\text{Spin}(8n)$-bundle corresponding to $E$, that is, we have a $G$-$\text{Spin}(8n)$-module $V$ such that $E \cong R \times_{\text{Spin}(8n)} V$ ($V$ is actually the fibre of $E$, at the base point of $X$, and the equivalence above follows from the fact that $X$ is $G$-connected).

As $\text{Spin}(8n)$ acts freely on $R$, we see that

$$KO_{G_{\text{Spin}(8n)}}(R) \cong KO_G(R/\text{Spin}(8n)) \cong KO_G(X),$$

and that

$$KO_{G_{\text{Spin}(8n)}}(R \times V) \cong KO_G(E) \cong KO_G(T(E))$$

as $E$ is not a compact $G$-space. The result follows now immediately from (5.1).

QED

We construct a $G$-Hopf-map $e : F/O_G \to BS_O^G$ as follows:

Let $X$ be a finite $G$-connected $G$-$\text{CW}$-complex. Then the elements in $[X,F/O_G]^G$ can be described as 3-tuples $(E,F,h)$, where $E$ and $F$ are stable $G$-bundles over $X$, such that $E - F \in KSO_G(X)$ and where $h$ is a fibrewise $G$-homotopy equivalence $h : S^E \to S^F$. (See [BM], p.146 for a closer description of the group structure on $[X,F/O_G]^G$.)

Since 2 is inverted, we can assume that $E$ and $F$ are $G$-$\text{Spin}$-bundles, and by stabilizing, we can further assume that $E$ and $F$ are $G$-$\text{Spin}(8n)$-bundles.

Let $\Delta_E = \Phi_E(1) \in KO_G(T(E))$ and $\Delta_F = \Phi_F(1) \in KO_G(T(F))$ be the Thom-classes of $E$ and $F$. $h$ gives a map $T(E) \to T(F)$, and we define $e(E,F,h)$ as the unique element in $1 + KO_G(X)$ satisfying

$$h^*(\Delta_F) = e(E,F,h) \cdot \Delta_E$$

– observe that $KO_G(T(E))$ is a free $KO_G(X)$-module of rank 1, and that $\Delta_E$ and $h^*(\Delta_F)$ are the image of units of $KO_G(X)$.
**Proposition 5.4**

We have a $G$-homotopy commutative diagram

\[
\begin{array}{cccc}
F/O_G & \longrightarrow & BSO_G^\oplus \\
e & \downarrow & \rho^k \\
BSO_G^\otimes & \underset{1/\psi^k}{\longrightarrow} & BSO_G^\otimes
\end{array}
\]

where $k$ is an integer, and $i : F/O_G \rightarrow BSO_G^\otimes$ is the 'inclusion' map.

**Proof:**

Let $X$ be a finite, $G$-connected $G$-CW-complex, $(E,F,h) \in [X,F/O_G]^G$. Then

\[
(1/\psi^k \circ e)(E,F,h) = (1/\psi^k) \left( \frac{h^*(\Delta_P)}{\Delta_E} \right) = \frac{\psi^k \Delta_E}{\Delta_E} \cdot \frac{h^*(\Delta_P)}{h^*(\psi^k \Delta_P)} = \rho^k (E) \cdot (\rho^k (F))^{-1} = \rho^k (E - F) = \rho^k (i(E,F,h))
\]

QED

**Corollary 5.5**

The composite $SF_G \longrightarrow F/O_G \longrightarrow BSO_G^\otimes$ factors through $J_G$.

**Proof:**

We must show that the composite $SF_G \longrightarrow F/O_G \longrightarrow BSO_G^\otimes \underset{1/\psi^k}{\longrightarrow} BSO_G^\otimes$ is nullhomotopic. But from (5.4) we have the homotopy commutative diagram

\[
\begin{array}{cccc}
SF_G & \longrightarrow & F/O_G & \longrightarrow & BSO_G^\otimes \\
e & \downarrow & \rho^k \\
BSO_G^\otimes & \underset{1/\psi^k}{\longrightarrow} & BSO_G^\otimes
\end{array}
\]

and as $i \circ j$ is null-homotopic, we get the result.

QED

**Lemma 5.6**

Let $k$ be as in (4.12). Let $\alpha_k : BSO_G^\otimes \rightarrow F/O_G$ be the map of (4.11). Then the composite $e \circ \alpha_k : BSO_G^\otimes \rightarrow BSO_G^\otimes$ is $G$-homotopic to $\rho^k : BSO_G^\otimes \rightarrow BSO_G^\otimes$.

**Proof:**

We have the diagram
Oriented, equivariant $K$-theory and the Sullivan splittings
Kenneth Hansen

\[ \begin{align*}
BSO_G^\oplus & \xrightarrow{1-\psi^k} BSO_G^\oplus \\
\alpha_k & \\
F / O_G & \xrightarrow{i} BSO_G^\oplus \\
\iota & \downarrow p^k \\
BSO_G^\oplus & \xrightarrow{1/\psi^k} BSO_G^\oplus
\end{align*} \]

which is homotopy commutative because of (4.11) and (5.4). As

\[ \begin{align*}
BSO_G^\oplus & \xrightarrow{1-\psi^k} BSO_G^\oplus \\
\rho^k & \\
BSO_G^\oplus & \xrightarrow{1/\psi^k} BSO_G^\oplus
\end{align*} \]

is commutative, too, we see that $1/\psi^k \circ (e \circ \alpha^k)$ and $1/\psi^k \circ \rho^k$ are $G$-homotopic maps.

As in [AII], p.152, it is possible to define $\rho^k$ on a complex $G$-bundle $E \xrightarrow{} X$ by using the Thom-isomorphism $\Phi_E : K_G(X) \rightarrow \bar{K}_G(T(E))$, where $T(E)$ is the Thom-complex of $E$, cf. [A], (4.8). We have

\[(5.7) \quad \rho^k(E) = \Phi_E^{-1} \circ \psi^k \circ \Phi_E(l) \in K_G(X),\]

and from [AII], (5.4), we get

\[(5.8) \quad \Phi_E^{-1} \circ \psi^k \circ \Phi_E(x) = \rho^k(E) \cdot \psi^k(x), \quad x \in K_G(X)\]

(This definition of $\rho^k$ coincides with that of [AT], p. 281 and p. 268 – see [AT], p.286 ff.).

Letting $Y = S^{2n} = T(\mathbb{C}^n \xrightarrow{} \ast)$ and by using the exponential nature of $\rho^k$ and its behaviour on complex line-bundles, we see that $\rho(\mathbb{C}^n \xrightarrow{} \ast) = k^n$ and from [tD], (3.5.1), and (5.8), we get

\[(5.9) \quad (\psi^k(\chi))(g) = k^n \cdot \chi(g), \quad g \in G,\]

where $\chi \in \bar{K}_G(S^{2n})$ is considered as a complex character under the Thom-isomorphism

$\Phi_{\psi^k} : R(G) = K_G(\ast) \rightarrow \bar{K}_G(S^{2n})$.

As 2 is inverted, the map

$KSO_G(S^{2n}) \cong RO(G) \rightarrow R(G) \cong \bar{K}_G(S^{2n})$

given by 'complexification' of representations, is injective, and preserves the $\lambda$-ringstructure.

Selecting a $\mathbb{Z}$-basis for $RO(G)$ consisting of the irreducible representations, we see that the matrix of the map $\psi^k - 1$ has non-vanishing determinant – modulo $k$ this matrix is simply the diagonal matrix with $-1$ as the only entries. We conclude that $\psi^k - 1$ induces monomorphisms

$\pi_{2n}((\psi^k - 1)^H) : \pi_{2n}(BSO_G^H) \rightarrow \pi_{2n}(BSO_G^H)$.
for every subgroup $H$ of $G$.

Going over to the multiplicative structure, we again have that $1/\psi^k$ gives monomorphisms in homotopy (for odd $n$, $\pi_n(BSO^n_H)$ vanishes). We conclude that $e \circ \alpha_k$ and $\rho^k$ give the same maps on the homotopy groups.

If we now consider $\rho^k$ and $e \circ \alpha_k$ as natural transformations between the representable functors $KSO_G(\cdot)$ and $1 + KSO_G(\cdot)$, we see that they coincide on the $G$-cells $S^n \wedge (G/H)_+$. We want to show that $\rho^k$ and $e \circ \alpha_k$ coincide on every $G$-CW-complex.

As $KSO_G(BSO_G)$ is torsion-free, ([MR], at the bottom of p. 97,) it suffices to show that $\rho^k$ and $e \circ \alpha_k$ coincide after rationalization. By applying (2.9), which states that both $BSO^G \otimes \mathbb{Q}$ and $BSO^G \otimes \mathbb{Q}$ are products of equivariant Eilenberg-MacLane-spaces, and Elmendorf's description of $G$-cohomology, [El], p.277, the problem reduces to show that for every integer $n > 2$ and subgroup $H$ of $G$ the natural transformations

$$H^n(\cdot; \pi_n(BSO^n_H) \otimes \mathbb{Q}) \rightarrow H^n(\cdot; \pi_n(BSO^n_H) \otimes \mathbb{Q})$$

induced by $\pi*((\rho^k)^n)$ and $\pi*((e \circ \alpha^k)^n)$ coincide. But $(\rho^k)^n$ and $(e \circ \alpha^k)^n$ agree on homotopy groups, and the result follows.

QED

**Definition 5.10**

Recall the $G$-Hopf-Space splitting

$$BSO^G \simeq B_0^\otimes \times (B_0^\otimes)\perp$$

of (3.9), where $(B_0^\otimes)\perp = B_1^\otimes \times \ldots \times B_{m-1}^\otimes$. Let $\pi$ and $\pi^\perp$ be the projections $\pi : BSO_G \rightarrow B_0^\otimes$ and $\pi^\perp : BSO_G \rightarrow (B_0^\otimes)\perp$.

Define $\beta : F/O_G \rightarrow BSO_G$ as the composite

$$F/O_G \xrightarrow{\Delta} F/O_G \times F/O_G \xrightarrow{e \times i} BSO^G \times BSO^G \xrightarrow{\pi \times \pi^\perp} BSO^G \otimes BSO^G \xrightarrow{\id \times \delta} BSO^G \otimes BSO^G \xrightarrow{\pi \times \pi^\perp} BSO^G \otimes BSO^G$$

Here $\Delta$ is the diagonal map, while $\delta$ is the map from (3.12).

Finally, define the $G$-space $\text{Cok}J_G$ as the homotopy fibre of $\beta$.

We are now able to generalize the splittings of Sullivan [MN, (5.18)] to the equivariant case.

**Theorem 5.11**

$\beta$ gives a splitting $F/O_G \simeq BSO_G \times \text{Cok}J_G$

**Proof:**

24
We show that $\beta \circ \alpha_k : BSO_G \rightarrow BSO_G$ is a $G$-homotopy equivalence:

$$\beta \circ \alpha_k : BSO_G \rightarrow BSO_G$$

G-homotopic to the composite

$$BSO_G \xrightarrow{\Lambda} BSO_G \times BSO_G \xrightarrow{\pi \times \pi \cdot \delta(y^{k-1})} B_0^\oplus \times (B_0^\oplus)^\perp$$

as it follows from (5.6) and (4.11). Separating $BSO_G^\oplus$ into $B_0^\oplus$ and $(B_0^\oplus)^\perp$, we see that the composite

$$B_0^\oplus \rightarrow BSO_G \xrightarrow{\beta \circ \alpha_k} B_0^\oplus \times (B_0^\oplus)^\perp$$

equals

$$B_0^\oplus \xrightarrow{\Lambda} B_0^\oplus \times B_0^\oplus \xrightarrow{\pi \times \pi \cdot \delta(y^{k-1})} B_0^\oplus \times (B_0^\oplus)^\perp$$

where 0 is a null-homotopic map, while the composite

$$(B_0^\oplus)^\perp \rightarrow BSO_G \xrightarrow{\beta \circ \alpha_k} B_0^\oplus \times (B_0^\oplus)^\perp$$

becomes

$$(B_0^\oplus)^\perp \xrightarrow{\Lambda} (B_0^\oplus)^\perp \times (B_0^\oplus)^\perp \xrightarrow{\pi \times \pi \cdot \delta(y^{k-1})} B_0^\oplus \times (B_0^\oplus)^\perp$$

Thus, if we separate the homotopy groups of the spaces $BSO_G^\oplus$ and $BSO_G^\otimes$ into direct summands $\pi_n(BSO_G^\oplus) = \pi_n(B_0^\oplus) \oplus \pi_n((B_0^\oplus)^\perp)$ and $\pi_n(BSO_G^\otimes) = \pi_n(B_0^\otimes) \oplus \pi_n((B_0^\otimes)^\perp)$, the matrix of $\beta \circ \alpha_k$ becomes

$$\begin{pmatrix}
\rho^k & \rho^k \\
0 & \delta(y^{k-1})
\end{pmatrix}$$

It suffices to show that $\rho^k : B_0^\oplus \rightarrow B_0^\oplus$ and $\delta(y^{k-1}) : (B_0^\oplus)^\perp \rightarrow (B_0^\oplus)^\perp$ are $G$-homotopy-equivalences. The first fact follows from (3.11), while the second is more or less obvious – one needs the fact that $\delta$ preserves the splittings (3.9) and (3.10), but this follows from the construction of $\delta$, (3.12) and [AS], thm. 3. Furthermore, on the factor $(B_0^\otimes)^\perp$, the map $\psi^k - 1 : (B_0^\otimes)^\perp \rightarrow (B_0^\otimes)^\perp$ is a $G$-homotopy-equivalence, as this follows from the proof of (3.9), and the description of $(B_0^\otimes)^\perp$ therein.

QED

**Corollary 5.12**

We have a splitting

$$SF_G = J_G \times \text{Cok}J_G$$

**Proof:**

We have the $G$-homotopy commutative diagram

$$
\begin{array}{ccc}
J_G^\oplus & \rightarrow & BSO_G^\oplus \\
\pi \downarrow & & \alpha \downarrow \\
SF_G & \rightarrow & F/O_G \\
\bar{\pi} \downarrow & & \bar{\beta} \downarrow \\
J_G^\otimes & \rightarrow & BSO_G^\otimes \\
\end{array}
$$

where $\bar{\beta}$ is the snap from (3.13). Here the horizontal sequences are fibration sequences, and the maps $\bar{\alpha}$ and $\bar{\beta}$ are the maps induced by $\alpha$ and $\beta$. 

25
Since $\beta \circ \alpha$ and $\delta$ are $G$-homotopy equivalences, a five-lemma argument on every fixed point set diagram for every subgroup $H$ of $G$ shows that $\delta \circ \alpha$ is a $G$-homotopy equivalence. As $\delta$ is a $G$-homotopy-equivalence, the homotopy fibres of $\beta$ and $\delta$ must be the same, namely $\text{Cok} J_g$.

QED

References


[K] K. Kawakubo: $\Lambda G$-Structure of $G$-Vector Bundles and Groups $KO_g(X)$, $KSp_g(X)$ and $J_g(X)$. Osaka J. Math, 19 (1982), 695-715.


